A METHOD TO SOLVE A PARTICULAR OPTIMAL CONTROL PROBLEM USED AS MODEL FOR AN ECONOMIC PROCESS

O METODĂ DE REZOLVARE A UNEI PROBLEME PARTICULARE DE CONTROL OPTIMAL, UTILIZATĂ CA MODEL AL UNUI PROCES ECONOMIC

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Abstract: *Description of the characteristics and their variations in an economic process using optimal control theory represent already classic problems in economic – mathematical modeling. There well know the difficulties which appear in determination of the solutions of such problems using classical methods and numeric. This paper presents the realization of a particular case production control model which depends of the speed variation of stocks; we assume the costs depend upon the production and the variation of production. We suppose also that on the market there is a constant demand, because the economic unit is small enough such that the production could not take effects on global demand. After presenting the model, the determination of the optimal solution is made using an approximate method.*

Key words*: Optimal control; production control model*

INTRODUCTION

We denote by u(t), $t \in [t_0, t_1]$, the production at the moment *t*, and let v(t), be the stock at the moment *t* (Kecs, W. (1981)). Let $v_0 = v(t_0)$ the initial stock at the moment t_0 and c(t) the demand at the moment *t*, and *f*, the cost per time units of production, more precisely, $f = f(t, v(t), u(t))$, It follows that the total cost is described by the functional:

(1.1) $I(u(t)) = \int_{t_0}^{t} f(x, v(x), u(x))dx$.

The problem is to find the production u(t) such that the demand c(t) is satisfied, f is minimum and of course the stock are minimum. The quantity of products which can be offered in the time interval $[t, t+\Delta t]$ is:

$c(t')\Delta t = \int_t^{t+\Delta t} c(x)dx$,

The quantity of products removed from the deposit is: $x(t) - x(t+\Delta t)$, and the quantity produced in this interval is, $\int_t^{t+\Delta t} u(x)dx = u(t'')\Delta t$, where, $t'' \in [t, t+\Delta t]$. The quantity the supply is the sum between the quantity produced and the quantity removed from the deposit. We have: $u(t'')\Delta t + x(t) - x(t+\Delta t) = c(t')\Delta t$. Taking the limit for $\Delta t \to 0$, we have $t' \to t$, $t'' \to t$ and it follows that, $u(t)dt - dv = c(t)dt$. Dividing by dt, we obtain:

(1.2) $$\frac{u(t)dt}{dt} - \frac{dv(t)}{dt} = \frac{c(t)dt}{dt} \Leftrightarrow -\frac{dv(t)}{dt} + u(t) = c(t).$$

We consider the model as an optimal control problem; we have to find the function u(t) solution of the differential equation (1.2) such that the functional (1.1) is minimum under condition $u \in [0, u_0]$, $x_0 \geq 0$, where u_0 represent the maximum of production (Pontriaquine, L. et all (1962, 1974)).

THE CONSTRUCTION OF THE MODEL OF A PRODUCTION PROCESS CORRESPONDING TO THE PARTICULAR CASE USING OPTIMAL CONTROL THEORY

We apply the above problem to a production unit which has a deposit under the hypothesis the variation of stock is proportional with the speed of the variation of production $u'(t)$.

Let $v(t) = a \cdot u'(t) + b$, a>1,b>0. We suppose that on the market there is a constant demand , $c(t) = k$, because the unit is small enough such that the production could not take effects on global demand. It follows the cost per time unit can be written:

$f(t, u(t), v(t)) = F(t, u(t), u'(t))$,

and the total costs of the production will be:

$$I[u(t)] = \int_{t_0}^{t} F(x, u(x), u'(x))dx .$$

The differential equation

$$(2.1)\quad -\frac{dv(t)}{dt} + u(t) = c(t)$$

becomes:

$$-\frac{d}{dt}[au'(t) + b] + u(t) = k$$

$$(2.2)\quad -au''(t) + u(t) = k$$

$$u''(t) - \frac{1}{a}u(t) = -\frac{k}{a}$$

a nonnonhomogenous second order differential equations with constant coefficients. To solve this equation, first we determine the general solution of the corresponding homogenous equation ($u_H(t)$):

$$(2.3)\quad u''(t) - \frac{1}{a}u(t) = 0$$

The characteristic equation is:

$$r^2 - \frac{1}{a} = 0 \Rightarrow r_1 = \sqrt{\frac{1}{a}}, r_2 = -\sqrt{\frac{1}{a}} .$$

The solutions are real and distinct; we have:

$$u_H(t) = c_1 e^{\sqrt{\frac{1}{a}}t} + c_2 e^{-\sqrt{\frac{1}{a}}t},\ c_1, c_2 \in \mathbf{R}.$$

In order to determinate a particular solution y_p, of the nonhomogenous equation, we use the undetermined coefficients. We have: $-\frac{k}{a} = e^{\tilde{\alpha}t}P_n(t)$. It follows that $\tilde{\alpha} = 0$, is not a solution of the characteristic equation and $P_n(t) = -\frac{k}{a}$ a polynomial of degree $n = 0$. We suppose that $u_p(t) = c,\ c \in \mathbf{R}$. Substituting in the equation (2.2) it follows:

$$\left[u_p(t)\right]'' - \frac{1}{a}u_p(t) = -\frac{k}{a}$$

$$c'' - \frac{1}{a}c = -\frac{k}{a} \Rightarrow c = k \Rightarrow u_p(t) = k.$$

The general solution of the equation (2.2) is: $u(t) = u_H(t) + u_P(t)$,

(2.4) $u(t) = c_1 e^{\sqrt{\frac{1}{a}}t} + c_2 e^{-\sqrt{\frac{1}{a}}t} + k,\ c_1, c_2, k \in \mathbf{R}$

Let $u(t) = c_1 u_1(t) + c_2 u_2(t) + k$, where: $u_1(t) = e^{\sqrt{\frac{1}{a}}t}$, $u_2(t) = e^{-\sqrt{\frac{1}{a}}t}$. We have:

$$u_1(t) = e^{\sqrt{\frac{1}{a}}t} \Rightarrow u_1(0) = e^{\sqrt{\frac{1}{a}}\cdot 0} = 1$$

$$u_1'(t) = \sqrt{\frac{1}{a}}e^{\sqrt{\frac{1}{a}}t} \Rightarrow u_1'(0) = \sqrt{\frac{1}{a}}e^{\sqrt{\frac{1}{a}}\cdot 0} = \sqrt{\frac{1}{a}}$$

$$u_1''(t) = \left(\sqrt{\frac{1}{a}}\right)^2 e^{\sqrt{\frac{1}{a}}t} \Rightarrow u_1''(0) = \left(\sqrt{\frac{1}{a}}\right)^2 e^{\sqrt{\frac{1}{a}}\cdot 0} = \left(\sqrt{\frac{1}{a}}\right)^2$$

$$u_1'''(t) = \left(\sqrt{\frac{1}{a}}\right)^3 e^{\sqrt{\frac{1}{a}}t} \Rightarrow u_1'''(0) = \left(\sqrt{\frac{1}{a}}\right)^3 e^{\sqrt{\frac{1}{a}}\cdot 0} = \left(\sqrt{\frac{1}{a}}\right)^3$$

...

$$u_1^{(n)}(t) = \left(\sqrt{\frac{1}{a}}\right)^n e^{\sqrt{\frac{1}{a}}t} \Rightarrow u_1^{(n)}(0) = \left(\sqrt{\frac{1}{a}}\right)^n e^{\sqrt{\frac{1}{a}}\cdot 0} = \left(\sqrt{\frac{1}{a}}\right)^n$$

Therefore,

$$u_1(t) = 1 + t\cdot\frac{\sqrt{\frac{1}{a}}}{1!} + t^2\cdot\frac{\left(\sqrt{\frac{1}{a}}\right)^2}{2!} + t^3\cdot\frac{\left(\sqrt{\frac{1}{a}}\right)^3}{3!} + \ldots + t^n\cdot\frac{\left(\sqrt{\frac{1}{a}}\right)^n}{n!} + \ldots$$

also,

$$u_2(t)=e^{-\sqrt{\frac{1}{a}}t} \Rightarrow u_2(0)=e^{-\sqrt{\frac{1}{a}}\cdot 0}=1$$

$$u_2'(t)=-\sqrt{\frac{1}{a}}e^{-\sqrt{\frac{1}{a}}t} \Rightarrow u_2'(0)=-\sqrt{\frac{1}{a}}e^{-\sqrt{\frac{1}{a}}\cdot 0}=-\sqrt{\frac{1}{a}}$$

$$u_2''(t)=\left(-\sqrt{\frac{1}{a}}\right)^2 e^{-\sqrt{\frac{1}{a}}t} \Rightarrow u_2''(0)=\left(-\sqrt{\frac{1}{a}}\right)^2 e^{-\sqrt{\frac{1}{a}}\cdot 0}=\left(-\sqrt{\frac{1}{a}}\right)^2$$

$$u_2'''(t)=\left(-\sqrt{\frac{1}{a}}\right)^3 e^{-\sqrt{\frac{1}{a}}t} \Rightarrow u_2'''(0)=\left(-\sqrt{\frac{1}{a}}\right)^3 e^{-\sqrt{\frac{1}{a}}\cdot 0}=\left(-\sqrt{\frac{1}{a}}\right)^3$$

...

$$u_2^{(n)}(t)=\left(-\sqrt{\frac{1}{a}}\right)^n e^{-\sqrt{\frac{1}{a}}t} \Rightarrow u_2^{(n)}(0)=\left(-\sqrt{\frac{1}{a}}\right)^n e^{-\sqrt{\frac{1}{a}}\cdot 0}=\left(-\sqrt{\frac{1}{a}}\right)^n$$

therefore,

$$u_2(t)=1-t\cdot\frac{\sqrt{\frac{1}{a}}}{1!}+t^2\cdot\frac{\left(\sqrt{\frac{1}{a}}\right)^2}{2!}-t^3\cdot\frac{\left(\sqrt{\frac{1}{a}}\right)^3}{3!}+\ldots+(-1)^n t^n\cdot\frac{\left(\sqrt{\frac{1}{a}}\right)^n}{n!}+\ldots$$

we have:

$$u(t)=k+c_1+c_2+t\left(c_1\frac{\sqrt{\frac{1}{a}}}{1!}-c_2\frac{\sqrt{\frac{1}{a}}}{1!}\right)+t^2\left(c_1\frac{\left(\sqrt{\frac{1}{a}}\right)^2}{2!}+c_2\frac{\left(\sqrt{\frac{1}{a}}\right)^2}{2!}\right)+R(t)$$

where R(t) represents a risk factor which is close to zero when a>1.

We shall return now to the cost of the production per time unit, $F(t,u(t),u'(t))$. The problem is to find the production u(t), such that the functional,

$$(2.5)\quad I[u(t)]=\int_{t_0}^{t} F(x,u(x),u'(x))dx$$

attains its minimum.

According to Euler's theorem (from variation calculus) the problem is to determinate a solution of the differential equation: $\frac{\partial F}{\partial u}-\frac{d}{dt}\left(\frac{\partial F}{\partial u'}\right)=0$, which satisfies $\frac{\partial^2 F}{\partial u'^2}>0$. Let suppose the function of the cost per time unit is:

$$F(t,u(t),u'(t))=\alpha u(t)+\beta[u'(t)]^2,\ \alpha,\beta>0,$$

which shows that the variation of production implies a rapidly variation of expenses.

We have:

$$I[u(t)]=\int_{t_0}^{t}F(x,u(x),u'(x))dx,\quad I[u(t)]=\int_{t_0}^{t}[\alpha u(x)+\beta(u'(x))^2]dx.$$

We shall solve the equation: $\frac{\partial F}{\partial u}-\frac{d}{dt}\left(\frac{\partial F}{\partial u'}\right)=0$. We have:

$$\frac{\partial F}{\partial u}=\alpha;\ \frac{\partial F}{\partial u'}=2\beta u'(t),\ \frac{\partial^2 F}{\partial u'^2}=2\beta>0$$

it follows:

$$\alpha-\frac{d}{dt}[2\beta u'(t)]=0\Rightarrow\alpha-2\beta u''(t)=0\Rightarrow u''(t)=\frac{\alpha}{2\beta}\ .$$

Using successive integrals, we obtain:

$$u'(t)=\frac{\alpha}{2\beta}t+c_3\Rightarrow u(t)=\frac{\alpha}{4\beta}t^2+c_3t+c_4,\ c_3,c_4\in\mathbf{R}.$$

In order to determine the solution of the optimal control problem we shall identify the coefficients of the two expression of the function u(t), obtained , first as solution of the differential equation: $-\frac{dv(t)}{dt}+u(t)=c(t)$

And then as function which realizes the minimum of the functional:

$$I[u(t)]=\int_{t_0}^{t}F(x,u(x),u'(x))dx.$$

The two equivalent expressions of u(t) are:

$$u(t) = k + c_1 + c_2 + t\left(c_1\frac{\sqrt{\frac{1}{a}}}{1!} - c_2\frac{\sqrt{\frac{1}{a}}}{1!}\right) + t^2\left(c_1\frac{\left(\sqrt{\frac{1}{a}}\right)^2}{2!} + c_2\frac{\left(\sqrt{\frac{1}{a}}\right)^2}{2!}\right)$$

$$u(t) = c_4 + c_3 t + \frac{\alpha}{4\beta}t^2,\ c_1, c_2, c_3, c_4 \in \mathbf{R}.$$

CONCLUSION

After the coefficients identification, we obtain the following linear system:

$$\begin{cases} \frac{\alpha}{2\beta} = \frac{c_1}{a} + \frac{c_2}{a} \\ c_3 = \frac{c_1}{\sqrt{a}} - \frac{c_2}{\sqrt{a}} \\ c_4 = k + c_1 + c_2 \end{cases}$$

For $c_3 = \gamma,\ \gamma \in \mathbf{R}$ we have:

$$\begin{cases} c_1 = \frac{\frac{a\alpha}{2\beta} + \gamma\sqrt{a}}{2} \\ c_2 = \frac{\frac{a\alpha}{2\beta} - \gamma\sqrt{a}}{2} \\ c_3 = \gamma \\ c_4 = k + \frac{a\alpha}{2\beta} \end{cases}.$$

It follows the optimal program (for a stock and a minimal cost) is given by:

$$u(t) = \frac{\alpha}{4\beta}t^2 + \gamma t + k + \frac{a\alpha}{2\beta},\ \gamma \in \mathbf{R},\ \alpha, \beta > 0.$$

Reducing the problem to the particular case presented above does not necessarily means a restriction of generality taking into account the multiple situations which could be described using this model.

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